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EXISTENCE FOR IMPLICIT DIFFERENTIAL EQUATIONS WITH NONMONOTONE PERTURBATIONS

BY

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ABSTRACT

This paper is devoted to the existence of solutions for a class of implicit Cauchy problems. The main tools in our study will be a convergent approximation procedure and the theory of pseudomonotone perturbations of maximal monotone mappings.

1. Introduction

Consider the implicit Cauchy problem

$$(1.1)\qquad \begin{cases} \frac{d}{dt}(Bu(t)) + Au(t) + Gu(t) = f, \text{ a.e. } t \in (0,T), \\ Bu(0) = Bu_0, \end{cases}$$

in a real Hilbert space V. Here B is a linear bounded, symmetric and positive operator from V to V' (the dual space of V), A is monotone and G is not.

Problem (1.1) allows many special cases that have already been studied. If $G = 0$, the existence of solutions for the implicit Cauchy problem (1.1) was recently obtained by Barbu and Favini in [2] using the theory of monotone operators. When B is the identity operator on V, (1.1) is the problem considered by Liu in [3] (where V can be a reflexive Banach space). More special cases can be found in the references of the papers cited above.

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Our purpose in this paper is to prove the existence of solutions for the implicit Cauchy problem (1.1) by using the theory of pseudomonotone perturbations of maximal monotone mappings.

2. Preliminaries

Let V, H be real Hilbert space. V' stands for the dual of V and $V \subseteq H \subseteq V'$ algebraically and topologically. The norm of any Banach space U is denoted by $\|\cdot\|_U$. The duality pairing between U and U' is denoted by $\langle\cdot,\cdot\rangle_U$. Let p, q and T be constants such that $T > 0, p \geq 2$ and $1/p + 1/q = 1$. Let $X = L^p(0,T;V), X' = L^q(0,T;V'), I = [0,T]$. The norm convergence is denoted by $\rightarrow$ and the weak convergence by $\rightharpoonup$.

Now, we introduce the following conditions.

(H_1) $B \in L(V,V'), \langle Bu, u\rangle_V \geq 0$ for all $u \in V$ and B is symmetric.

Here $L(V,V')$ denotes the space of all bounded linear operators from V into V'.

(H_2) A: $V \rightarrow V'$ is monotone and semicontinuous.

(H_3) G: $V \rightarrow V'$ is both continuous and weakly continuous. Furthermore, for any sequence $\{u_n\}$ in V with $u_n \rightharpoonup u$ in V, we have

$$\limsup \langle Gu_n, u_n - u\rangle_V \geq 0.$$

(H_4) There exist positive constants c_1, c_2, c_3 and c_4 such that

$$\begin{aligned} \|Au\|_{V'} &\leq c_1(\|u\|_V^{p-1} + 1), \\ \|Gu\|_{V'} &\leq c_2(\|u\|_V^{p-1} + 1), \\ \langle Au + Gu, u\rangle_V &\geq c_3\|u\|_V^p - c_4, \ \forall u \in V. \end{aligned}$$

3. Main results

Let Λ: $V \rightarrow V'$ be the canonical isomorphism and $\epsilon > 0$ be given. Under assumption (H_1), we see that $\epsilon\Lambda + B$ is an isomorphism from V to V'. So we can let

$$\langle u, v\rangle_W := \langle u, (\epsilon\Lambda + B)^{-1}v\rangle_V \quad \forall u, v \in V'.$$

Since B is symmetric, it is easy to show that $\langle\cdot,\cdot\rangle_W$ is an inner product on V' and the space $W := (V', \langle\cdot,\cdot\rangle_W)$ is a Hilbert space in which the norm is denoted by $\|\cdot\|_W$. And the two norms on V' are equivalent, i.e.,

$$\|(\epsilon\Lambda + B)^{-1}\|^{-1/2}\|v\|_W \leq \|v\|_{V'} \leq \|\epsilon\Lambda + B\|^{1/2}\|v\|_W \quad \forall v \in V'. \tag{3.1}$$

In fact, let $v \in V'$. Then

$$\begin{aligned}\|v\|_W^2 =&\langle v, (\epsilon\Lambda + B)^{-1}v\rangle_V\\ \leq&\|v\|_{V'}\|(\epsilon\Lambda + B)^{-1}v\|_V\\ \leq&\|(\epsilon\Lambda + B)^{-1}\|\|v\|_{V'}^2,\end{aligned}$$

which implies the first part of our inequalities. Also, there exists $u \in V$, $\|u\|_V = 1$ such that $\|v\|_{V'} = \langle v, u\rangle_V$. Write $z = (\epsilon\Lambda + B)u \in V'$.

$$\begin{aligned}\|v\|_{V'} =&\langle v, u\rangle_V\\ =&\langle v, (\epsilon\Lambda + B)^{-1}z\rangle_V\\ =&\langle v, z\rangle_W\\ \leq&\|v\|_W\|z\|_W\\ =&\|v\|_W\langle z, z\rangle_W^{1/2}\\ =&\|v\|_W\langle z, (\epsilon\Lambda + B)^{-1}z\rangle_V^{1/2}\\ =&\|v\|_W\langle z, u\rangle_V^{1/2}\\ \leq&\|v\|_W\|z\|_{V'}^{1/2}\|u\|_V^{1/2}\\ \leq&\|v\|_W\|\epsilon\Lambda + B\|^{1/2}\|u\|_V.\end{aligned}$$

Since $\|u\| = 1$, we have

$$\|v\|_{V'} \leq \|\epsilon\Lambda + B\|^{1/2}\|v\|_W.$$

Therefore (3.1) holds.

We assume that $f \in X', u_0 \in V$, and $(H_1) - (H_4)$ hold. Let $Z = L^p(0, T; W)$. Since W is a Hilbert space, identifying W with its dual, we may write $Z' = L^q(0, T; W)$ (see [4, 412] for details). Define $A_\epsilon, G_\epsilon \colon Z \to Z'$ as

$$\begin{aligned}(A_\epsilon v)(t) &= A((\epsilon\Lambda + B)^{-1}v + u_0)(t) - f(t),\\ (G_\epsilon v)(t) &= G((\epsilon\Lambda + B)^{-1}v + u_0)(t).\end{aligned}$$

Then we have

LEMMA 3.1: *Suppose that the assumptions $(H_1) - (H_4)$ hold. Then for each $u_0 \in V, f \in X'$ and $\epsilon > 0$ the mapping $A_\epsilon + G_\epsilon \colon Z \to Z'$ is coercive, bounded, semicontinuous, and pseudomonotone.*

Proof: Since $(\epsilon\Lambda + B)^{-1}$ is a bounded linear operator, by $(H_2) - (H_3)$ and (3.1) we easily obtain that the sum operator $A_\epsilon + G_\epsilon$ is bounded and semicontinuous.

Now we shall first prove that $A_\epsilon + G_\epsilon$ is coercive. In fact

$$\begin{aligned}
&\langle (A_\epsilon + G_\epsilon)u, u\rangle_W \\
=&\langle A((\epsilon\Lambda + B)^{-1}u + u_0) + G((\epsilon\Lambda + B)^{-1}u + u_0) - f(t), (\epsilon\Lambda + B)^{-1}u\rangle_V \\
\geq& c_3\|(\epsilon\Lambda + B)^{-1}u + u_0\|_V^p - c_4 - \|f(t)\|_{V'}\|(\epsilon\Lambda + B)^{-1}u\|_V \\
&- \|u_0\|_V(c_1 + c_2)(\|(\epsilon\Lambda + B)^{-1}u + u_0\|_V^{p-1} + 1) \\
\geq& c_3\|(\epsilon\Lambda + B)^{-1}\|^p\|u\|_{V'}^p - \|f(t)\|_{V'}\|(\epsilon\Lambda + B)^{-1}\|\|u\|_{V'} \\
&- c_5\|(\epsilon\Lambda + B)^{-1}\|^{p-1}\|u\|_{V'}^{p-1} - c_6,
\end{aligned}$$

where c_5 and c_6 are independent of u. Since the two norms $\|\cdot\|_{V'}$ and $\|\cdot\|_W$ on V' are equivalent by (3.1), we have

$$\langle (A_\epsilon + G_\epsilon)u, u\rangle_W \geq C_1\|u\|_W^p - C_2\|f(t)\|_{V'}\|u\|_W - C_3\|u\|_W^{p-1} - C_4,$$

which implies that

$$\langle (A_\epsilon + G_\epsilon)u, u\rangle_Z \geq C_1'\|u\|_Z^p - C_2'\|f\|_{X'}\|u\|_Z - C_3'\|u\|_Z^{p-1} - C_4',$$

where C_1', C_2', C_3', C_4' are positive constants. Since $p \geq 2$, from the above inequality we obtain

$$\frac{\langle (A_\epsilon + G_\epsilon)u, u\rangle_Z}{\|u\|_Z} \to \infty$$

as $\|u\|_Z \to \infty$, which proves the coercivity of $A_\epsilon + G_\epsilon$.

At last we shall show that $A_\epsilon + G_\epsilon$ is pseudomonotone. Suppose $v_n \rightharpoonup v$ in Z and

$$\limsup \langle A_\epsilon v_n + G_\epsilon v_n, v_n - v\rangle_Z \leq 0. \tag{3.2}$$

Let $u_n = (\epsilon\Lambda + B)^{-1}v_n, u = (\epsilon\Lambda + B)^{-1}v$. Since $(\epsilon\Lambda + B)^{-1}$ is a bounded linear operator, it is also weakly continuous, which implies that $u_n \rightharpoonup u$ in X. We shall prove the pseudomonotonicity of $A_\epsilon + G_\epsilon$ by showing

$$\langle A_\epsilon v_n + G_\epsilon v_n, w\rangle_Z \to \langle A_\epsilon v + G_\epsilon v, w\rangle_Z \quad \forall w \in Z \tag{3.3}$$

and

$$\liminf \langle A_\epsilon v_n + G_\epsilon v_n, v_n\rangle_Z = \langle A_\epsilon v + G_\epsilon v, v\rangle_Z. \tag{3.4}$$

Define

$$h_n(t) = \langle A_\epsilon v_n(t) + G_\epsilon v_n(t), v_n(t) - v(t)\rangle_W \quad \forall t \in I.$$

First we show that for all $t \in I$,

$$\liminf h_n(t) \geq 0. \tag{3.5}$$

Suppose that the assertion is false, that is, for some $t_0 \in I$,

$$\liminf h_n(t_0) < 0. \tag{3.6}$$

It follows from (H_4) that

$$\begin{aligned} h_n(t_0) =& \langle A_\epsilon v_n(t_0) + G_\epsilon v_n(t_0), v_n(t_0) - v(t_0)\rangle_W \\ =& \langle A_\epsilon v_n(t_0) + G_\epsilon v_n(t_0), (\epsilon\Lambda + B)^{-1}(v_n(t_0) - v(t_0))\rangle_V \\ =& \langle A((\epsilon\Lambda + B)^{-1}v_n + u_0)(t_0) - f(t_0) + G((\epsilon\Lambda + B)^{-1}v_n + u_0)(t_0), \\ & (\epsilon\Lambda + B)^{-1}(v_n(t_0) - v(t_0))\rangle_V \\ =& \langle A(u_n + u_0)(t_0) - f(t_0) + G(u_n + u_0)(t_0), u_n(t_0) - u(t_0))\rangle_V \\ \geq& c_3\|u_n(t_0) + u_0\|_V^p - c_4 - \|f(t_0)\|_{V'}\|u_n(t_0) - u(t_0)\|_V \\ & - (c_1 + c_2)(\|u_n(t_0) + u_0\|_V^{p-1} + 1)\|u(t_0) + u_0\|_V. \end{aligned} \tag{3.7}$$

By (3.6) and (3.7) we get that $\{u_n(t_0)\}$ is bounded in V. Therefore, passing to a subsequence if necessary we can assume that $u_n(t_0) \rightharpoonup u(t_0)$ in V. Using assumption (H_3), we obtain

$$\liminf\langle G(u_n + u_0)(t_0), u_n(t_0) - u(t_0)\rangle_V \geq 0.$$

It follows from the monotonicity of A and the weak convergence of $u_n(t_0)$ that

$$\liminf\langle A(u_n + u_0)(t_0) - f(t_0), u_n(t_0) - u(t_0)\rangle_V \geq 0.$$

Hence, we have

$$\begin{aligned} \liminf h_n(t_0) \geq& \liminf\langle A(u_n + u_0)(t_0) - f(t_0), u_n(t_0) - u(t_0)\rangle_V \\ & + \liminf\langle G(u_n + u_0)(t_0), u_n(t_0) - u(t_0))\rangle_V \\ \geq& 0, \end{aligned}$$

which contracts (3.6) and thereby proves (3.5). Then by Fatou's Lemma, we obtain from (3.2)

$$\begin{aligned} 0 \leq& \int_0^T \liminf h_n(t)dt \\ \leq& \liminf \int_0^T h_n(t)dt \end{aligned}$$

$$\begin{aligned}&\leq \limsup \int_0^T h_n(t)dt\\ &= \limsup \langle A_\epsilon v_n + G_\epsilon v_n, v_n - v)\rangle_Z\\ &\leq 0.\end{aligned}$$

The inequalities above imply that $\liminf h_n(t) = 0$. Passing to a subsequence if necessary, we have

$$\lim_{n\to\infty} h_n(t) = 0 \quad \text{a.e. } t \in I. \tag{3.8}$$

Since (3.7) holds for all $t \in I$, it follows from (3.8) that $\{u_n(t)\}$ is bounded in V. Therefore, $\{v_n(t)\}$ is bounded in W for a.e. $t \in I$, and hence we may assume $v_n(t) \rightharpoonup v(t)$ in W for almost all $t \in I$. Since G is weakly continuous, and the linear bounded operator $(\epsilon\Lambda + B)\colon V \to V'$ is, of course, weakly continuous, we have that for any $w \in Z$,

$$\langle (G_\epsilon v_n)(t) - (G_\epsilon v)(t), w(t)\rangle_W \to 0 \quad \text{a.e. } t \in I.$$

Using the boundedness of $\{v_n\}$, and (H_4) and (3.1), we obtain

$$\begin{aligned}&|\langle (G_\epsilon v_n)(t) - (G_\epsilon v)(t), w(t)\rangle_W|\\ &\leq \|G((\epsilon\Lambda + B)^{-1}v_n + u_0)(t) - G((\epsilon\Lambda + B)^{-1}v + u_0)(t)\|_W \|w(t)\|_W\\ &\leq (k_1\|v(t)\|_W^{p-1} + k_2)\|w(t)\|_W,\end{aligned}$$

where k_1, k_2 are positive constants.

By the Dominated Convergence Theorem, it follows from the above facts that

$$\langle G_\epsilon v_n - G_\epsilon v, w\rangle_Z = \int_0^T \langle (G_\epsilon v_n)(t) - (G_\epsilon v)(t), w(t)\rangle_W dt \to 0 \quad \forall w \in Z,$$

i.e.,

$$G_\epsilon v_n \rightharpoonup G_\epsilon v \quad \text{in } Z'. \tag{3.9}$$

Similarly, using $(H_3), (H_4)$ and (3.1), we have

$$\begin{aligned}|\langle (G_\epsilon v_n)(t), v_n(t) - v(t)\rangle_W| &\leq \|(G_\epsilon v_n)(t)\|_W \|v_n(t) - v(t)\|_W\\ &\leq k_3\|v(t)\|_W + k_4,\end{aligned}$$

and

$$\liminf \langle (G_\epsilon v_n)(t), v_n(t) - v(t)\rangle_W \geq 0.$$

Using Fatou's Lemma again, we obtain

$$\begin{aligned} 0 &\leq \int_0^T \liminf \langle (G_\epsilon v_n)(t), v_n(t) - v(t) \rangle_W dt \\ &\leq \liminf \int_0^T \langle (G_\epsilon v_n)(t), v_n(t) - v(t) \rangle_W dt, \end{aligned}$$

i.e.,

$$\liminf \langle G_\epsilon v_n, v_n \rangle_Z \geq \langle G_\epsilon v, v \rangle_Z. \tag{3.10}$$

So from (3.2) and (3.10), we have

$$\limsup \langle A_\epsilon v_n, v_n - v \rangle_Z \leq 0.$$

By the definition of A_ϵ, we readily get that A_ϵ is a monotone and semicontinuous operator on the real reflexive Banach space Z. It follows from Proposition 32.7 in [4] that A_ϵ is maximal monotone. Therefore, using Lemma 1.3 of Chap. II in [1], we have

$$A_\epsilon v_n \rightharpoonup A_\epsilon v \tag{3.11}$$

and

$$\lim_{n\to\infty} \langle A_\epsilon v_n, v_n \rangle_Z = \langle A_\epsilon v, v \rangle_Z. \tag{3.12}$$

From (3.2) and (3.11), (3.12), we obtain

$$\limsup \langle G_\epsilon v_n, v_n - v \rangle_Z \leq 0.$$

So by (3.10), we get

$$\lim_{n\to\infty} \langle G_\epsilon v_n, v_n - v \rangle_Z = 0 \tag{3.13}$$

Hence from (3.9)–(3.13) we obtain (3.3) and (3.4), which proves our Lemma 3.1.

Now we are in a position to obtain our main results.

THEOREM 3.2: *Let $f \in X'$ and $u_0 \in V$ be given. Under assumptions $(H_1)-(H_4)$, problem (1.1) has at least one solution $u \in X$ such that $Bu \in L^p(0,T;V')$, $(Bu)' \in L^q(0,T;V')$.*

Proof: For $\epsilon > 0$, consider the approximating equation

$$\begin{cases} ((\epsilon\Lambda + B)u(t))' + Au(t) + Gu(t) = f, \text{ a.e. } t \in I, \\ u(0) = u_0. \end{cases} \tag{3.14}$$

We can rewrite (3.14) as

$$\begin{cases} v'(t) + A_\epsilon v(t) + G_\epsilon v(t) = 0, \text{ a.e. } t \in I, \\ v(0) = 0. \end{cases} \tag{3.15}$$

Define $Lv = v'$ and $D(L) = \{v \in Z : v' \in Z', v(0) = 0\}$. Here v' stands for the generalized derivative of v, i.e.,

$$\int_0^T v'(t)\phi(t)dt = -\int_0^T v(t)\phi'(t)dt \quad \forall \phi \in C_0^\infty(I).$$

It can be shown (see Proposition 32.10 of [4]) that L is a maximal monotone mapping.

By means of the operator L, we can rewrite (3.15) as

$$Lv + A_\epsilon v + G_\epsilon v = 0, \quad v \in D(L). \tag{3.16}$$

In virtue of Theorem 32.A of [4] and our Lemma 3.1, we obtain that equation (3.16) has a solution $v_\epsilon \in D(L)$ for any $\epsilon > 0$, which implies problem (3.14) has a solution $u_\epsilon \in X$ and $u_\epsilon' \in X'$.

Now we write equation (3.14) as

$$\begin{cases} ((\epsilon\Lambda + B)u_\epsilon(t))' + Au_\epsilon(t) + Gu_\epsilon(t) = f, \text{ a.e. } t \in I, \\ u_\epsilon(0) = u_0. \end{cases} \tag{3.17}$$

Multiplying (3.17) by u_ϵ, we get (after using (H_4))

$$\begin{aligned} &\frac{\epsilon}{2}\frac{d}{dt}\langle \Lambda u_\epsilon(t), u_\epsilon(t)\rangle_V + \frac{1}{2}\frac{d}{dt}\langle Bu_\epsilon(t), u_\epsilon(t)\rangle_V + c_3\|u_\epsilon(t)\|_V^p \\ &\le \|f(t)\|_{V'}\|u_\epsilon(t)\|_V + c_4 \quad \text{a.e. } t \in I. \end{aligned}$$

Integrating the last inequality on $(0,t)$, $t > 0$, we get

$$\begin{aligned} &\frac{\epsilon}{2}\|u_\epsilon(t)\|_V^2 + \frac{1}{2}\langle Bu_\epsilon(t), u_\epsilon(t)\rangle_V + c_3\int_0^t \|u_\epsilon(s)\|_V^p ds \\ \le &\left(\int_0^t \|f(s)\|_{V'}^2 ds\right)^{1/2}\left(\int_0^t \|u_\epsilon(s)\|_V^2 ds\right)^{1/2} + c_4 T + \frac{\epsilon}{2}\|u_0\|_V^2 + \frac{1}{2}\langle Bu_0, u_0\rangle_V. \end{aligned}$$

By Young's inequality, we obtain

$$\epsilon\|u_\epsilon(t)\|_V^2 + \langle Bu_\epsilon(t), u_\epsilon(t)\rangle_V + \int_0^t \|u_\epsilon(s)\|_V^p ds \le C, \tag{3.18}$$

where the constant C may depend on $\|f\|_{X'}$ and $\|u_0\|_V$.

By assumption (H_4) and (3.18) we know that $\{u_\epsilon\}$ is bounded in X, $\{Au_\epsilon\}$ and $\{Gu_\epsilon\}$ are bounded in X'. Hence on a subsequence, again denoted $\{\epsilon\}$, we have

$$u_\epsilon \rightharpoonup u \quad \text{in } X, \tag{3.19}$$

$$Au_\epsilon \rightharpoonup x \quad \text{in } X', \tag{3.20}$$

$$Gu_\epsilon \rightharpoonup y \quad \text{in } X', \tag{3.21}$$

$$Bu_\epsilon \rightharpoonup Bu \quad \text{in } X', \tag{3.22}$$

$$((\epsilon\Lambda + B)u_\epsilon)' \rightharpoonup (Bu)' \quad \text{in } X'. \tag{3.23}$$

In view of (3.19)–(3.23), to conclude the proof of existence it remains only to show that $x = Au$ and $y = Gu$ a.e., $t \in I$.

Now scalar multiplying (3.17) by $(u_\epsilon - u)$ and integrating, we have

$$\begin{aligned}&\langle Au_\epsilon + Gu_\epsilon, u_\epsilon - u\rangle_X \\ =&\langle f, u_\epsilon - u\rangle_X - \langle[(\epsilon\Lambda + B)(u_\epsilon - u)]', u_\epsilon - u\rangle_X - \langle[(\epsilon\Lambda + B)u]', u_\epsilon - u\rangle_X.\end{aligned}$$

By (3.19), we obtain

$$\begin{aligned}&\limsup\langle Au_\epsilon + Gu_\epsilon, u_\epsilon - u\rangle_X \\ &\le \limsup\Big\{ -\frac{\epsilon}{2}\|u_\epsilon(T) - u(T)\|_V^2 - \frac{1}{2}\langle B(u_\epsilon(T) - u(T)), u_\epsilon(T) - u(T)\rangle_V\Big\} \\ &\le 0.\end{aligned}$$

Similar to the proof of (3.3), we easily get

$$Au_\epsilon \rightharpoonup Au, \quad Gu_\epsilon \rightharpoonup Gu,$$

i.e., $x = Au$ and $y = Gu$, which complete the proof of Theorem 3.2. ∎

Remark: In our previous paper [3], we dealt with evolution variational inequalities with nonmonotone perturbations. As a special case of the variational inequalities, i.e., the closed convex subset M is the whole space X in the main theorem of [3], we obtained an existence result in [3] for the corresponding evolution equations, which also follows from Theorem 3.2 relative to $B =$ *identity*.

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